

# Multistring solutions in inflationary spacetimes

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## Abstract

Multistring solutions of the string equations of motion are found for inflationary spacetimes with expansion factor  $R(\eta) \propto \eta^k$  for any  $k < 0$  and  $\eta$  conformal time. If  $0 > k > -1$  only two-string solutions can be found within our ansatz, whereas for  $k = -1 - 1/n$ , with  $n = 1, 2, \dots$ , multistring solutions exist with an infinite number of strings [In the special case  $k = -2$  we recover de Sitter spacetime where multistring solutions were first found].

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## I. INTRODUCTION AND MOTIVATIONS

The study of string dynamics in curved spacetime reveals new insights and new physical phenomena with respect to string propagation in flat spacetime [1]- [2]. The results of this programme are relevant both for fundamental (quantum) strings and for cosmic strings, which behave essentially in a classical way.

Recently, a novel feature for strings in de Sitter spacetime was found: exact multi-string solutions [3]- [4]. The novel feature is that one single world-sheet gives multiple (infinitely many) strings. The world-sheet time  $\tau$  turns out to be a multiply-valued [3] (and even an infinite-valued [4]) function of the target string time  $X^0$  (which can be the comoving or cosmic time  $t$  or the conformal time  $\eta$ ). Each branch of  $\tau$  as a function of  $X^0$  corresponds to a different string. In flat spacetime, multiple string solutions are necessarily described by multiple world-sheets. Here, a single world-sheet describes many different and simultaneous strings as a consequence of the coupling with the spacetime geometry. These strings do not interact among themselves; all the interaction is with the curved spacetime.

In the present note we show that multistring solutions are present in universes with scale factor  $R(\eta) \propto \eta^k$  for  $k$  a real number. [Here  $\eta$  stands for conformal time]. Within our ansatz the multistring solutions for  $0 > k > -1$  are shown to contain at most two strings, as opposed to the case  $k < -1$ , for which the possibility of having an infinite number of strings exists.

## II. EQUATIONS OF MOTION.

Let us consider loop strings moving in cosmological background with metric

$$ds^2 = R^2(\eta) (d\eta^2 - d\vec{X}^2) ,$$

where  $\eta$  is the conformal time,  $\vec{X}$  stands for the vector of spatial coordinates, and  $R(\eta)$  is the conformal factor, which we assume takes the form  $R^2(\eta) = \eta^k$ . For  $k > 0$  we have FRW spacetimes:  $k = 4$  is the matter-dominated universe,  $k = 2$  is the radiation-dominated universe. For  $k < 0$  we have inflationary spacetimes:  $-2 < k < 0$  corresponds to superinflation,  $k = -2$  is de Sitter spacetime and  $k < -2$  corresponds to power inflation.

The following Ansatz for the string coordinates

$$\begin{aligned} \eta &= \eta(\tau) , \\ X^1 &= f(\tau) \cos \sigma , \\ X^2 &= f(\tau) \sin \sigma , \\ X^3 &= 0 . \end{aligned} \tag{1}$$

separates the string equations of motion and string constraints [5,6]. Here,  $\tau$  and  $\sigma$  stand for the worldsheet coordinates. That is, a set of ordinary differential equations follows for  $\eta(\tau)$  and  $f(\tau)$ :

$$\begin{aligned} \eta \ddot{\eta} + k \dot{f}^2 &= 0 , \\ \eta \ddot{f} + k \dot{\eta} \dot{f} + \eta f &= 0 , \\ \dot{\eta}^2 - \dot{f}^2 - f^2 &= 0 . \end{aligned} \tag{2}$$

The string proper size under this Ansatz becomes  $S(\tau) = \eta^{k/2}f$ . The string energy and pressure are given by

$$E(\tau) = \eta^{k/2}\dot{\eta} \quad , \quad P = \frac{\eta^{k/2}}{2\dot{\eta}}(\dot{f}^2 - f^2) \quad , \quad (3)$$

when we set the string tension to 1.

Examining equations (2), we observe that the third one (the constraint) is a conserved quantity of the first two. Furthermore, the system is autonomous. To top it all, the system is homogeneous in  $\eta, f$ . This means that we can reduce eqs. (2) to one single nonlinear differential equation, plus two quadratures [7,6].

Let us make the change of variables

$$z := \frac{f}{\eta} \quad , \quad v(z) := \frac{\dot{f}}{\dot{\eta}} \quad , \quad (4)$$

which leads to the equation

$$z(v - z)\frac{dv}{dz} + (1 + kzv)(1 - v^2) = 0 \quad . \quad (5)$$

Note that for the de Sitter universe ( $k = -2$ ),  $z$  is the proper size of the string, and  $v$  is given in terms of proper size and energy as  $v = S + \dot{S}/E$ .

For general  $k$ , we have

$$v^2 = 1 - \frac{S^2}{E^2} \quad ,$$

$$v = -\frac{k}{2}z + \frac{\dot{S}}{E} \quad .$$

The change of variables can be inverted as

$$\eta(z) = \eta(z_0) \exp \left( \int_{z_0}^z \frac{dz}{v(z) - z} \right) \quad , \quad (6)$$

$$f(z) = z\eta(z)$$

$$\dot{z}^2 = \frac{(v - z)^2 z^2}{1 - v^2} \quad . \quad (7)$$

From this last equation we immediately see that in order to have a physical solution  $v^2$  must be smaller than one. This requirement can be avoided if we allow for imaginary  $\tau$ . The significance of such a relaxation is that we would be admitting instantonic (or superluminal) strings. That is to say, string solutions such that the worldsheet metric has signature  $(-, -)$ , as opposed to the lorentzian one  $(+, -)$ .

A symmetry of equations (5) and (7) is given by the simultaneous change  $z \rightarrow -z$  and  $v \rightarrow -v$ . We can therefore concentrate on either  $z > 0$  or  $z < 0$ . We shall take  $z > 0$ . The physical strip is then

$$\{(z, v) \mid z \geq 0 \quad \text{and} \quad -1 < v < 1\} \quad . \quad (8)$$

From equation (5) we see that there will be singular points in the  $(z, v)$  plane given by the intersection of the lines

$$v = -\frac{1}{kz}, \quad \text{and} \quad v = \pm 1,$$

with the lines

$$v = z, \quad \text{and} \quad z = 0.$$

That is to say, the singular points will be

$$(0, \pm 1), \quad (\pm 1, \pm 1), \quad \left(\pm \frac{1}{\sqrt{-k}}, \pm \frac{1}{\sqrt{-k}}\right),$$

The last two singular points are real only for  $k < 0$ , and only for  $k < -1$  is one of them to be found in the physical strip. They correspond to exact solutions of eqs. (2) which were called instantons in Ref. [5].

In fig. 1 we portray the physical strip for de Sitter spacetime ( $k = -2$ ). The saddle point at  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  is the singular point corresponding to the instanton, which was called  $q^o$  in Ref. [3]. The physical strip for the case  $k = -3/2$  is portrayed in fig. 2. In the next figure, fig. 3, only the separatrices for de Sitter are shown.

Each singular point corresponds to asymptotic or exact solutions of eq. (2), as was displayed in Ref. [7]. We shall now analyze these asymptotics in order to know whether the singular points are reached in finite string time ( $\tau$ ). Recall that the presence of multistring solutions arises from the multivaluedness of  $\tau(\eta)$ . Since, if  $\tau(\eta)$  is multivalued, it follows that the physical range of  $\eta$ , that is,  $(0, \infty)$  must correspond to a finite range in  $\tau$ . Since it is clear that only the aforementioned singular points can correspond to the singular values of  $\eta$ , it follows that a necessary precondition for the presence of multistring configurations is that some of the graphs of the solutions to eq. (5) be given by a finite string time.

Note that the string time  $\tau$  is obtained from the solutions of eqs. (5) through (7) as

$$\tau - \tau_0 = \int_{z_0}^{z(\tau)} dz \frac{(1 - v^2)^{1/2}}{(v - z)z}. \quad (9)$$

### III. BEHAVIOUR NEAR THE SINGULAR POINTS

Let us briefly summarize the behaviour of  $\eta(\tau)$ ,  $f(\tau)$  and  $v(z)$  near the singular points  $(z, v) = (0, \pm 1)$ ,  $(\pm 1, \pm 1)$ ,  $(\pm \frac{1}{\sqrt{-k}}, \pm \frac{1}{\sqrt{-k}})$  [5] - [7] - [8].

The points  $(z, v) = (0, \pm 1)$  correspond to the string behaviour

$$\begin{aligned} \eta(\tau) &\stackrel{\tau \rightarrow \tau_0}{\approx} \eta_0 + \frac{\eta_0}{a\sqrt{2}}(\tau - \tau_0) + O(\tau - \tau_0)^3, \\ f(\tau) &\stackrel{\tau \rightarrow \tau_0}{\approx} \pm \frac{\eta_0}{a\sqrt{2}}(\tau - \tau_0) + O(\tau - \tau_0)^3, \\ v &\stackrel{z \rightarrow 0}{\approx} \pm(1 - az^2), \\ P &\stackrel{\tau \rightarrow \tau_0}{\approx} \frac{1}{2} E \stackrel{\tau \rightarrow \tau_0}{\approx} \frac{\eta_0^{k/2+1}}{2a\sqrt{2}}. \end{aligned} \quad (10)$$

where  $a > 0$ .

The relation between  $P$  and  $E$  is like in the case of radiation. It is similar to the dual to unstable behaviour although here  $P$  and  $E$  tend to *finite* constants. Eqs.(10) hold for all values of  $k$ .

Near the point  $(z, v) = (1, 1)$  we find

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow \tau_1}{\equiv} (\tau - \tau_1)^{1/(k+1)} \left[ 1 + \frac{k}{6} (\tau - \tau_1)^2 + O(\tau - \tau_1)^3 \right], \\
f(\tau) &\stackrel{\tau \rightarrow \tau_1}{\equiv} (\tau - \tau_1)^{1/(k+1)} \left[ 1 - \frac{k^2 + 2k + 3}{6(2k + 3)} (\tau - \tau_1)^2 \right. \\
&\quad \left. + O(\tau - \tau_1)^3 \right], \\
v &\stackrel{z \rightarrow 1}{\equiv} 1 + (2k + 3)(z - 1), \text{ for } k > -3/2 \\
v &\stackrel{z \rightarrow 1}{\equiv} 1 - A(1 - z)^{-2k-2}, \text{ for } k < -3/2 \\
P &\stackrel{\tau \rightarrow \tau_1}{\equiv} \frac{1}{2} E \stackrel{\tau \rightarrow \tau_1}{\equiv} \frac{1}{2(k+1)} (\tau - \tau_1)^{-\frac{k}{2(k+1)}}.
\end{aligned} \tag{11}$$

where  $A > 0$ .

We find here dual to unstable behaviour for  $k > 0$  and for  $k < -1$ . For  $-1 < k < 0$ , the equation of state is the same (radiation) but  $P$  and  $E$  tend to zero.

Near  $(z, v) = (\frac{1}{\sqrt{-k}}, \frac{1}{\sqrt{-k}})$ , we find for  $k < -1$ ,

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow +\infty}{\equiv} e^{\mp \sqrt{-k-1}\tau} \\
f(\tau) &\stackrel{\eta^{\pm 1} \rightarrow 0}{\equiv} \frac{\eta}{\sqrt{-k}} + a \eta^{\lambda_{\pm}} \\
v &\stackrel{z \rightarrow \frac{1}{\sqrt{-k}}}{\equiv} \frac{1}{\sqrt{-k}} + \lambda_{\pm} \left( z - \frac{1}{\sqrt{-k}} \right) \\
P &\stackrel{\tau \rightarrow \tau_1}{\equiv} \frac{1}{2} E \stackrel{\tau \rightarrow \tau_1}{\equiv} \frac{1}{2\sqrt{-k-1}} \eta^{k/2+1}
\end{aligned} \tag{12}$$

Here,  $\lambda_{\pm} \equiv -\frac{k}{2} \pm \frac{\sqrt{k^2 - 4k - 4}}{2}$ . We find here dual to unstable behaviour for  $k < -2$ ,  $\eta \rightarrow 0$  and for  $-2 < k < -1$ ,  $\eta \rightarrow \infty$ . Otherwise, the equation of state is the same (radiation) but  $P$  and  $E$  tend to zero.

Let us now consider the asymptotic points  $(z, v) = (+\infty, \pm 1)$  and  $(z, v) = (\infty, 0)$ .

We find for  $(+\infty, \pm 1)$  and  $k > 0$ ,

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow \tau_2}{\equiv} (\tau - \tau_2)^{1/(k+1)}, \\
f(\tau) &\stackrel{\tau \rightarrow \tau_2}{\equiv} f_0 \pm (\tau - \tau_2)^{1/(k+1)}, \\
v &\stackrel{z \rightarrow +\infty}{\equiv} \pm \left[ 1 - \frac{a}{z^k} \right] \\
P &\stackrel{\tau \rightarrow \tau_2}{\equiv} \frac{1}{2} E \stackrel{\tau \rightarrow \tau_2}{\equiv} \frac{1}{2(k+1)} (\tau - \tau_2)^{-\frac{k}{2(k+1)}} \rightarrow \infty.
\end{aligned} \tag{13}$$

where  $a > 0$ . Dual to unstable behaviour appears again.

For  $(+\infty, 0)$ , where the separatrices end for  $k > 0$  and  $k < -1$ , we have

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow \tau_3}{\equiv} \tau - \tau_3, \\
f(\tau) &\stackrel{\tau \rightarrow \tau_3}{\equiv} 1 - \frac{(\tau - \tau_3)^2}{2(k+1)},
\end{aligned}$$

$$\begin{aligned}
v &\stackrel{z \rightarrow \pm\infty}{=} -\frac{1}{(k+1)z} \rightarrow 0, \\
P &\stackrel{\tau \rightarrow \tau_3}{=} -\frac{1}{2}E \stackrel{\tau \rightarrow \tau_3}{=} -\frac{1}{2}(\tau - \tau_3)^{k/2}.
\end{aligned} \tag{14}$$

We find here the equation of state of unstable strings. However,  $P$  and  $E$  tend to zero for  $k > 0$  instead of blowing up.

When  $k$  is in the interval  $-1 < k < 0$  a different behaviour appears near  $(+\infty, 0)$ ,

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow \tau_3}{=} \tau - \tau_3, \\
f(\tau) &\stackrel{\tau \rightarrow \tau_3}{=} 1 + b(\tau - \tau_3)^{1-k} \\
v &\stackrel{z \rightarrow \pm\infty}{=} \pm a z^k \\
P &\stackrel{\tau \rightarrow \tau_3}{=} -\frac{1}{2}E \stackrel{\tau \rightarrow \tau_3}{=} -\frac{1}{2}(\tau - \tau_3)^{k/2} \rightarrow -\infty
\end{aligned} \tag{15}$$

Here we find unstable behaviour.

In Eqs.(10)-(15),  $\tau_0, \dots, \tau_3$  are arbitrary real constants.

For any value of  $k$  and large  $\tau$ , the ring strings may exhibit the oscillatory behaviour [5]

$$\begin{aligned}
\eta(\tau) &\stackrel{\tau \rightarrow \pm\infty}{=} \tau^{2/(k+2)}, \\
f(\tau) &\stackrel{\tau \rightarrow \pm\infty}{=} \frac{2}{k+2} \tau^{-k/(k+2)} \cos(\tau + \varphi),
\end{aligned} \tag{16}$$

where  $\varphi$  is a constant phase and the oscillation amplitude has been normalized. For large  $\tau$ , the energy and pressure of the solution (16) show stable behaviour

$$\begin{aligned}
E(\tau) &\stackrel{\tau \rightarrow \pm\infty}{=} \frac{2}{k+2} = \text{constant} \quad , \\
P(\tau) &\stackrel{\tau \rightarrow \pm\infty}{=} -\frac{1}{k+2} \cos(2\tau + 2\varphi) \rightarrow 0 \quad .
\end{aligned} \tag{17}$$

In the  $(z, v)$  variables this corresponds to

$$\begin{aligned}
z &\stackrel{\tau \rightarrow \pm\infty}{=} \frac{2}{k+2} \frac{\cos(\tau + \varphi)}{\tau} \\
v &\stackrel{\tau \rightarrow \pm\infty}{=} -\sin(\tau + \varphi) \quad .
\end{aligned} \tag{18}$$

This represents cycles wound around the segment  $-1 < v < +1$  with a width in the  $z$  direction of order

$$1/[\tau(1 + k/2)].$$

#### IV. PHASE PORTRAIT

We shall divide this section according to the bifurcation structure of eq. (5), i.e.,  $k > 0$ ,  $0 > k > -1$ , and  $-1 > k$ .

### A. $k > 0$

For  $k > 0$  there are three kinds of solutions in the physical strip, to which two separatrices are to be added. Namely, solutions joining

- i)  $(+\infty, -1)$  with  $(0, -1)$ ;
- ii)  $(+\infty, +1)$  with  $(0, -1)$ ;
- iii)  $(0, -1)$  with  $(0, 1)$ ,

and the separatrices from  $(0, -1)$  to  $(1, 1)$  and from  $(0, -1)$  to  $(+\infty, 0)$ . From the asymptotic behaviour of  $v(z)$  at the singular points, we see that solutions of type (i) and (ii) require a finite interval of  $\tau$ . That is, a finite  $\tau = \tau_i$  corresponds to the initial situation  $(+\infty, \pm 1)$ , and after a finite string time  $\tau_f$  we are to find ourselves at the point  $(0, -1)$ .

These solutions correspond to strings that originate at the big bang [ $R = 0, \eta = 0, (z, v) = (+\infty, \pm 1)$ ], that collapse to zero size at  $(0, -1)$  in a finite cosmological time. The posterior oscillatory behaviour of the worldsheet is given by type (iii) solutions, for which the  $\tau$  interval is also finite. The initial, type (i) or (ii) behaviour give rise to a unique type (iii) solution: equation (5) is singular at the point  $(0, -1)$ . Therefore, the solution is not unique: it is not determined by the initial value  $v(0)$ . As a matter of fact, it follows from the equation that for consistency  $v'(0) = 0$ . So the solutions to the equation are characterized by the second derivative of  $v$  with respect to  $z$  at the point  $z = 0$ :  $v \sim -1 + az^2$  [see eq.(10)]. For the physical reasons previously mentioned, it is clear that  $a > 0$ . After some manipulations, we can relate  $a$  to physical quantities, namely the scale factor at this point,  $R_0$ , and the energy of the string,  $E_0$ , at this same instant, by use of the expression

$$a = \frac{R_0^{1+2/k}}{2E_0^2}.$$

It follows that the solutions which reach this point are uniquely characterized by this physical quantity (here, by solutions we mean solutions in the  $(z, v)$  plane, which can each be mapped to a class of equivalent physical solutions for  $\eta$  and  $f$ ). Moreover, by the simultaneous mapping  $v \rightarrow -v$  and  $z \rightarrow -z$ , we can identify the solution in the physical strip that corresponds to an ingoing one with the calculated  $a$ , and we see that for the solutions coming from  $(+\infty, -1)$  to the point  $(0, -1)$  the continuing solution is of the oscillatory form iii).

Therefore, no multistring behaviour with infinite number of strings is found. The reason being that all the solutions previously stated require a semiinfinite interval of string time, thus making it impossible to have more than two strings present in each solution. Furthermore, in order to have even these two strings, it would be necessary to tie them together at  $\eta = 0$ . But the behaviour (13) forbids this; from which it follows that within our ansatz no multistring solutions exist for  $k > 0$ .

Let us now analyze the exceptional solutions, the separatrices. Both of them traverse a finite  $\tau$  interval, and both of them correspond to strings originating at the big bang which collapse later on. The difference with the previously mentioned type (i) and (ii) solutions is that whereas solutions (i) and (ii) exhibit dual to unstable behaviour [see eq.(13)] and

$S \sim R \rightarrow 0$  near  $(+\infty, \pm 1)$ , for the  $(1, 1)$  to  $(0, -1)$  separatrix there is also dual to unstable behaviour [see eq.(11)] near  $(1, 1)$  but there  $S \sim R^{1+2/k} \rightarrow 0$ . For the other one,  $(+\infty, 0)$  to  $(0, -1)$ , there is unstable behaviour near  $(+\infty, 0)$  and  $S \sim R \sim 1/E$  as  $R \rightarrow 0$ .

### B. $0 > k > -1$

There are two kinds of solutions in the physical strip:  $(0, -1)$  joined with  $(+\infty, 0)$ , and  $(0, -1)$  joined to  $(0, 1)$ . Additional to these, one finds a separatrix  $(0, -1)$  to  $(1, 1)$ . The  $\tau$  interval for the  $(0, -1)$  to  $(+\infty, 0)$  solutions is finite, and the strings thus described correspond to [see eq.(15)]  $R \sim S \sim E \sim z^{-k/2} \rightarrow \infty$  as  $z \rightarrow +\infty$ , with  $z \sim 1/(\tau - \tau_3)$ . For these superinflationary spacetimes, there are then strings that start at  $t = 0$ , [ $R = \infty$ ] and proceed to collapse, then continuing with an oscillatory motion (the second class of solutions), similarly to what happens in the previous  $k > 0$  case. The separatrix is also analogous to the previous case, and no multistring solutions with an infinite number of strings are present in the ring solutions to the string equations of motion for these spacetimes. Nonetheless, the argument against the existence of two-string solutions valid for  $k > 0$  fails in the case at hand, and two-string solutions exist within our ansatz for  $0 > k > -1$ . This is depicted in figs. 4. This is due to the fact that the behaviour for  $\eta \rightarrow 0$  is now given by eqns. (15).

### C. $k < -1$

There are four kinds of solutions to equation (5) in the physical strip, and four corresponding separatrices. The solutions join

- i)  $(+\infty, 0)$  to  $(1, 1)$ ,
- ii)  $(1, 1)$  to  $(0, 1)$ ,
- iii)  $(0, -1)$  to  $(0, 1)$ ,
- iv)  $(0, -1)$  to  $(+\infty, 0)$ ,

with separatrices from the point  $(1/\sqrt{-k}, 1/\sqrt{-k})$  to each of the points  $(0, \pm 1)$ ,  $(1, 1)$ , and  $(+\infty, 0)$ .

The  $(+\infty, 0)$  point corresponds to  $R \sim S \sim E \sim z^{-k/2} \rightarrow \infty$ , [see eq.(14)] and is reached in finite  $\tau$  time from the non-singular points connected to it. The points  $(0, \pm 1)$ , as before, correspond to finite  $R$  and  $E$ , and  $S = 0$ . The points  $(1, 1)$  and  $(1/\sqrt{-k}, 1/\sqrt{-k})$ , however, are more complicated.

As was mentioned before, the point  $(1/\sqrt{-k}, 1/\sqrt{-k})$  is a solution of the equations of motion of the string that we had previously called the instanton for  $k + 1 > 0$  [5]. For this solution, invariant string size and string energy are both proportional to cosmological time.

The point  $(1, 1)$  is a node for  $k < -1$ . Writing eq. (5) as a system of two differential equations, we obtain the linearized solution close to this point:

$$z \sim 1 - \frac{1-v}{2k+3} + \alpha (1-v)^{-1/(2k+2)}, \quad (19)$$



with  $\alpha$  some constant. It should be noticed that for  $k = -3/2$  this linearized solution involves a logarithmic term. From this equation, or from a direct analysis of eq. (5) we see that the leading behaviour of  $\eta(\tau)$  is given by  $(\tau - \tau_0)^{1/(k+1)}$ . The value  $\infty$  is reached by  $\eta$  in a finite time from the non-singular points connected to  $(1, 1)$ .

It is then clear that the interval  $(0, \infty)$  of  $\eta$  is obtained from a finite interval of string time, this being the prerequisite for the presence of multistring configurations. The point  $\eta = 0$  corresponds to the singular point  $(1/\sqrt{-k}, 1/\sqrt{-k})$  and to  $(z, v) = (+\infty, 0)$ . Disregarding the separatrix for the moment, consider a string that comes out of  $\eta = 0$  with positive  $df/d\eta$  (i.e., from  $(z, v) = (+\infty, 0^+)$ ). By following the evolution in the physical strip, we see it eventually reaches  $(z, v) = (1, 1)$ , in finite string time.

Similarly if the starting point is  $(z, v) = (+\infty, 0^-)$ : In its evolution in the  $(z, v)$  plane, the string moves towards the point  $(0, -1)$ . By the usual identification, this is the same as the point  $(0, 1)$ , from which it comes out, and then either goes on to an oscillatory behaviour, as before, or is driven to the point  $(1, 1)$ , which is again reached in a finite time.

In what concerns the point  $(1/\sqrt{-k}, 1/\sqrt{-k})$ , it can only be reached via the separatrices. It should be observed that it is a saddle point, and there are then two separatrices corresponding to  $\eta \rightarrow 0$  as we approach this singular point, and another two such that  $\eta \rightarrow \infty$ . So, if we were to start from  $(z, v) = (\infty, 0)$ , we would reach the point  $(1/\sqrt{-k}, 1/\sqrt{-k})$  through the more “horizontal” separatrices (either going through the points  $(0, \pm 1)$  or not). This would happen in infinite string time. On the other hand, the separatrix connecting the singular point  $(1/\sqrt{-k}, 1/\sqrt{-k})$  with  $(1, 1)$  corresponds to  $\eta = 0$  as we approach  $(1/\sqrt{-k}, 1/\sqrt{-k})$ , and  $\eta \rightarrow \infty$  as we approach  $(1, 1)$ . This also needs infinite string time  $\tau$  to take place.

Notice that an infinite string time  $\tau$  is needed to reach the point  $(1/\sqrt{-k}, 1/\sqrt{-k})$ . This is the only point which such property.

## V. MULTISTRING SOLUTIONS

The requirement for a multistring configuration is that both  $\eta(\tau)$  and  $X^0(\tau)$  be functions of  $\tau$  such that

1. the only singularities correspond to the spacetime singularity;
2. the infinite range of  $\eta$  ( $X^0$ ) is obtained from a finite range of string time.

These statements are well behaved under both worldsheet and spacetime diffeomorphisms.

In principle, the very fact that a finite  $\tau$  interval is mapped into the whole physical range allows us to write multistring configurations, and we have already shown some two-string solutions. For instance, in fig. 5 we portray a sequence of graphics for the case  $k = -1.6$ . First comes  $|\eta|$  as a function of  $\tau$ . Note in this respect that only the absolute value of  $\eta$  is relevant here; and, furthermore, that eqs. (2) are invariant under the mapping  $\eta \rightarrow -\eta$ . It should also be remarked that since the interval in  $\tau$  is finite, more strings can be accommodated onto the same worldsheet. Fig. 5b shows  $f$  as a function of  $\eta$  for this solution: it is clear that at least two strings are present at the same time. Both strings start

at  $\eta = 0$ . Their radius is the same only at  $\eta = 0$ , as depicted. To give a more complete picture, we next display  $v(z)$  for this solution.

The question remains posed, whether something similar to what has been achieved in de Sitter can be transposed to the more general context. In order to understand the problem better we shall study the results of [3], [4], translated into our formulation. As we have already pointed out, in de Sitter spacetime  $z$  is the proper size of the string, and  $v = \sqrt{1 - S^2/E^2}$ , so we can use the expressions given in [3] to obtain solutions to our equation (5). In particular, the hyperbolic solutions  $q_-$  and  $q_+$  ([3]) are given by

$$v_{II}(z) = \frac{2z - 1}{2z^2 - 2z + 1}. \quad (20)$$

When we make use of the invariance under  $z \rightarrow -z$ ,  $v \rightarrow -v$ , we see that this solution corresponds exactly to the separatrices in the physical strip, as shown in fig. 3, by considering also the solution

$$v_I(z) = \frac{2z + 1}{2z^2 + 2z + 1}. \quad (21)$$

Notice that  $z = \frac{1}{\sqrt{2}} \coth \frac{\tau}{\sqrt{2}}$  for the solution  $q_-$  and  $z = \frac{1}{\sqrt{2}} \tanh \frac{\tau}{\sqrt{2}}$  for the solution  $q_+$ . Therefore,  $z \geq \frac{1}{\sqrt{2}}$  corresponds to  $q_-$  and  $z \leq \frac{1}{\sqrt{2}}$  to  $q_+$ .

These two solutions, taken together, describe a total of three strings. One string solution corresponds to starting at the point  $(1/\sqrt{2}, 1/\sqrt{2})$  and following  $v_{II}$  downwards and to the left until it reaches  $(0, -1)$ , whence, by the usual mapping, we jump to  $(0, 1)$  and follow  $v_I$  until the point  $(1/\sqrt{2}, 1/\sqrt{2})$  is reached from the left. This requires infinite string time. Another string solution is given by the rest of  $v_{II}$ . That is starting at the point  $(1/\sqrt{2}, 1/\sqrt{2})$  upwards until the point  $(1, 1)$  is reached (in infinite string time), and then continuing by the line joining  $(1, 1)$  to  $(+\infty, 0)$ , which needs finite time to be traversed. A further string corresponds to the separatrix from  $(+\infty, 0)$  to the point  $(1/\sqrt{2}, 1/\sqrt{2})$ , requiring infinite time, and which is given by  $v_I$ .

The primitive that corresponds to  $(1 - v_{I,II}^2)^{(1/2)}/z(v_{I,II} - z)$  is  $\sqrt{2} \operatorname{arctanh}(\sqrt{2}z)$ , whence the previous statements follow. Notice that for  $z > 1/\sqrt{2}$  the free integration constant must be a complex number.

In [4] another class of multistring solutions is analyzed. Their explicit expressions in terms of elliptic functions are not illuminating for our present purposes, but their asymptotic behaviour close to the point  $(1, 1)$  in the  $(z, v)$  plane can be extracted, thus becoming amenable to comparison with eq. (19). The result is that  $v(z) \sim 1 - \beta^2(1 - z)^2 + \dots$  close to  $z = 1$ . On comparison with eq. (19) we see it tallies perfectly, as it should. We then conclude that for this type of multistring solutions to be present, we need some way of continuing solutions in the  $(z, v)$  plane through the singular point  $(1, 1)$ .

Notice that this demand is not fulfilled by the already depicted two-string solutions, and that this way of obtaining/identifying multistring solutions is different from the continuation through  $z = \infty$  (i.e.  $\eta = 0$ ) previously considered.

We can obtain multistring solutions by continuing solutions in the  $(z, v)$  plane through the singular point  $(1, 1)$ ; that means computing a solution of eq. (5) which is well behaved through the point  $(1, 1)$ , and then use eqs. (6) and (9) with a judicious choice of integration constants for the two branches of the  $(z, v)$  solution which come out of the point  $(1, 1)$ . In

order to do this consistently it is required that there be continuous, well-behaved solutions in the  $(z, v)$  plane through the singular point. From the leading behaviour in eqn. (19) we see that a necessary condition for this construction to work is that  $k = -1 - n$ : for these spacetimes multistring solutions exist with more than two strings.

Another alternative for the consistent construction of multistring solutions is given from the examination of eq.(11), whence we see that  $\eta$  and  $f$  are real near  $(1, 1)$  in both sides of the point  $\tau = \tau_1$  provided  $1/(k + 1)$  is an integer. Hence we find real multistring solutions for  $k = -1 - 1/n$  where  $n$  is a natural number. For  $n = 1$  we recover de Sitter spacetime.

In addition, we find from eq.(11) near the point  $\tau = \tau_1$ ,

$$v \stackrel{\tau \rightarrow \tau_1}{\approx} 1 - \frac{1}{2}(k + 1)^2 (\tau - \tau_1)^2 + O(\tau - \tau_1)^3 \quad (22)$$

This shows that the string always stays within the physical strip  $-1 \leq v \leq +1$  when  $\tau$  goes beyond  $\tau_1$ . In summary, an extra string appears for each point  $\tau_1$  exhibiting the behaviour (11) near  $(1, 1)$ .

The conclusion holds for generic solutions of the equations of motion, so the statement is even stronger than the mere existence of multistring solutions: for spacetimes with  $k = -1 - 1/n$  there exists a whole continuous class of multistring solutions to the (classical) string equations of motion.

The conclusion is that multistring solutions will always be allowed for spacetimes with scalar factor of the form  $R(\eta) \propto \eta^k$  with  $k < 0$ . For  $k < 0$  two-string solutions will be possible. For  $k = -1 - 1/n$  an infinite number of strings can be present in multistring solutions. For  $k = -1 - n$  multistring solutions are allowed with more than two strings. Note that de Sitter spacetime is included in both series of special spacetimes.

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## FIGURES

FIG. 1. Physical strip of the phase portrait of equation (5) for de Sitter spacetime.

FIG. 2. Physical strip of the phase portrait of equation (5) for  $k = -3/2$ .

FIG. 3. Separatrices for de Sitter spacetime

FIG. 4. Two-string solution for  $k = -0.5$ : a)  $|\eta|$  as a function of  $\tau$ ; b)  $f$  as a function of  $\eta$ ; c)  $v$  as a function of  $z$ .

FIG. 5. Multistring solution for  $k = -1.6$ : a)  $|\eta|$  as a function of  $\tau$ ; b)  $f$  as a function of  $\eta$ ; c)  $v$  as a function of  $z$ .

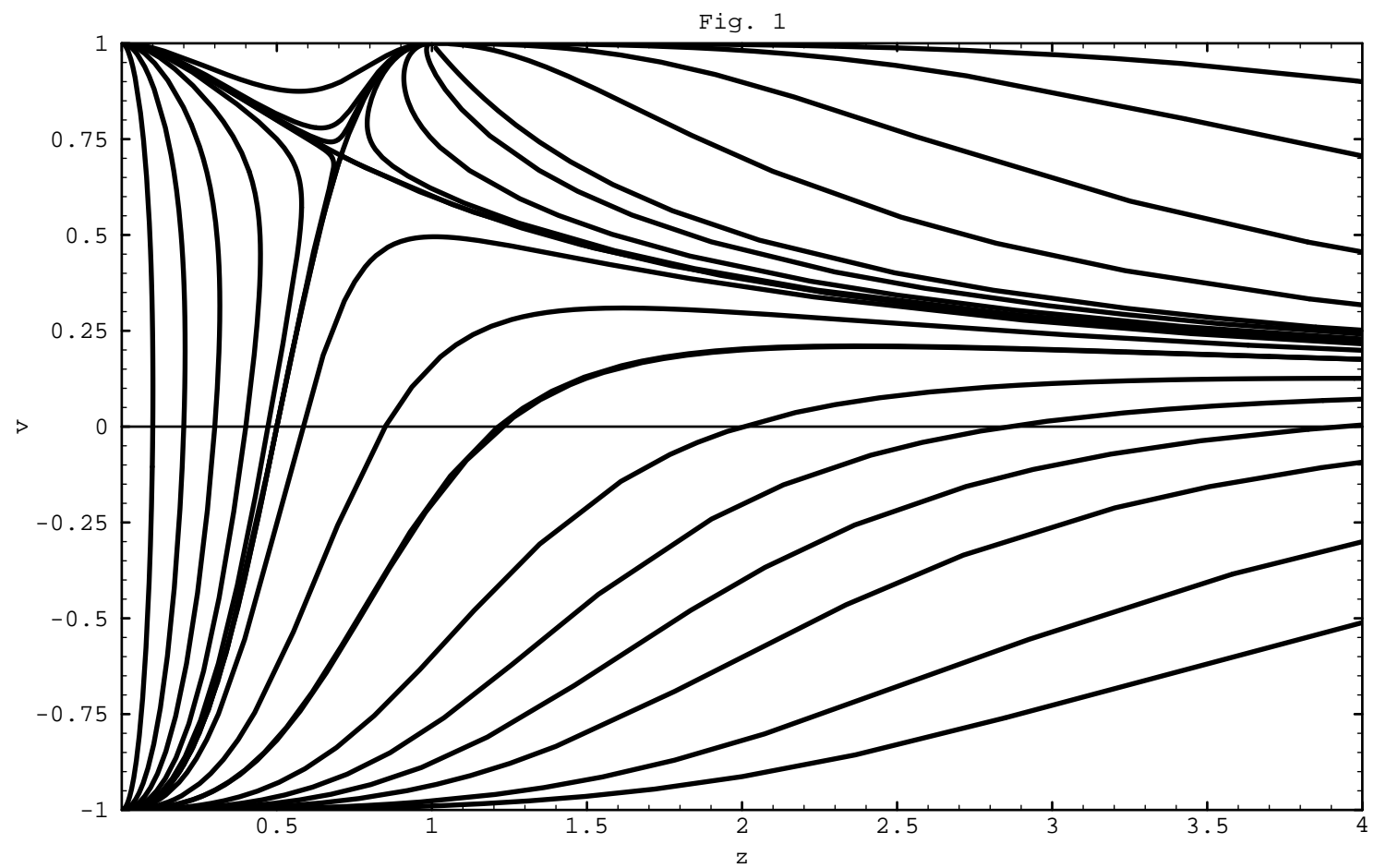


Fig. 2

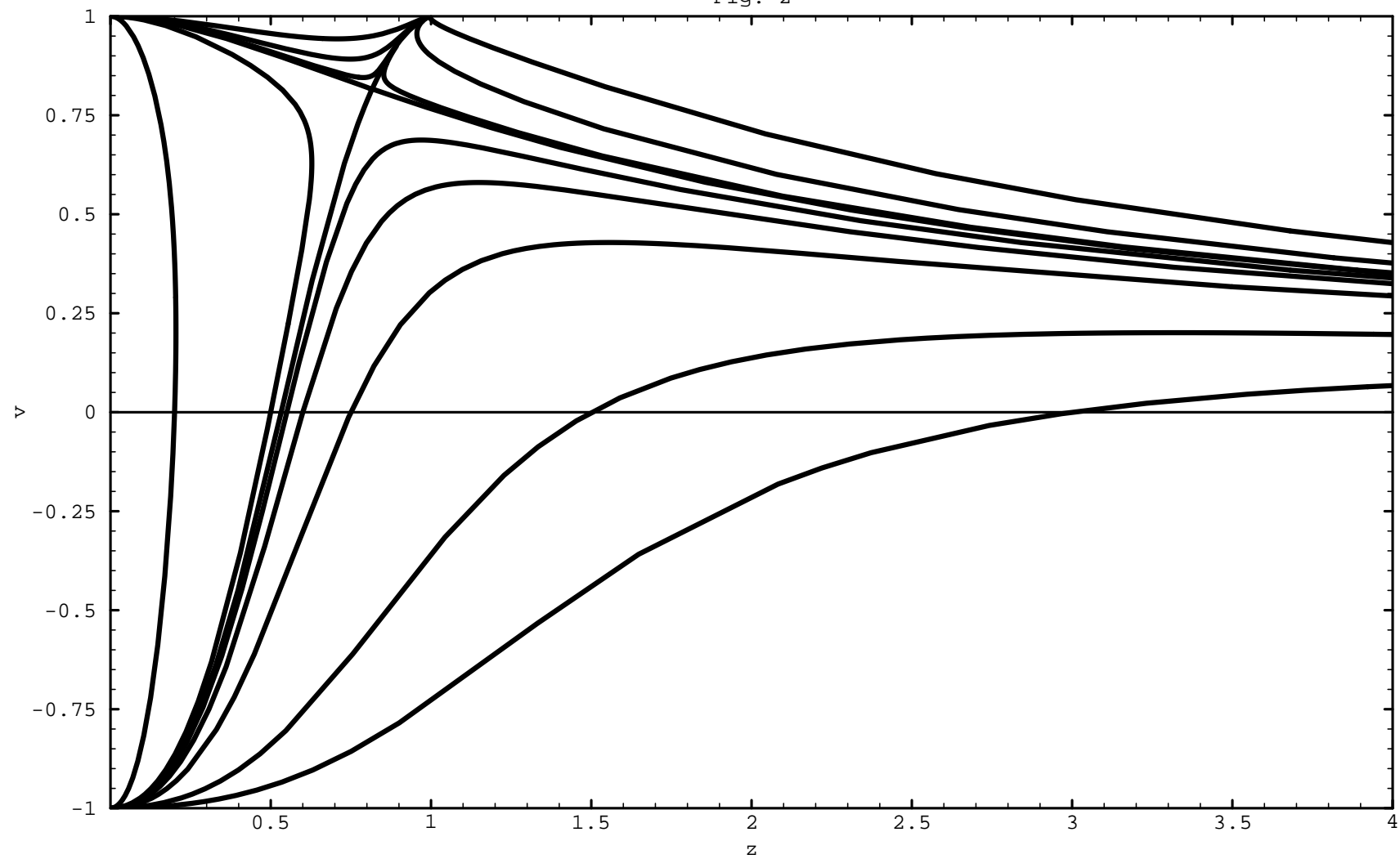


Fig. 3

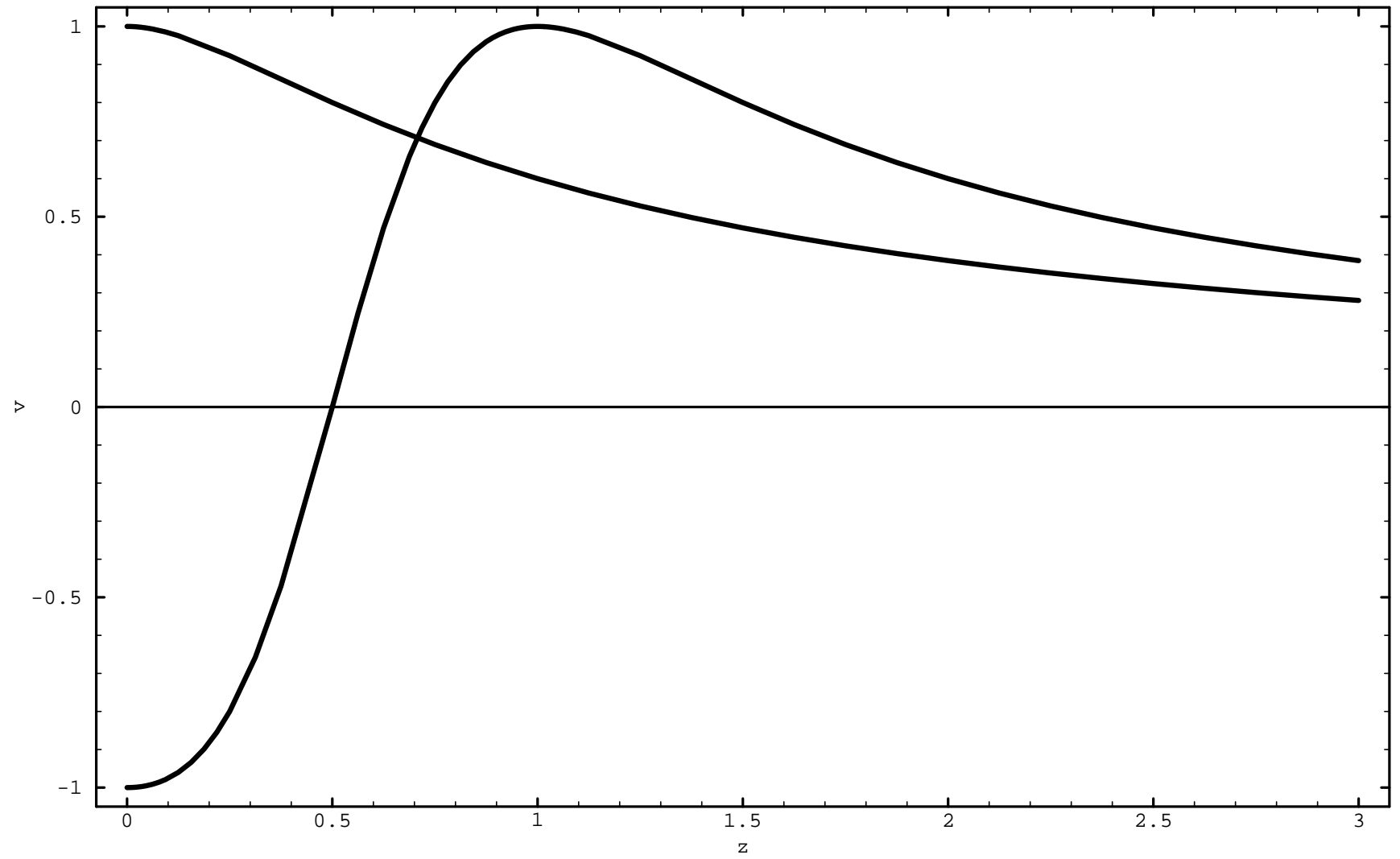




Fig. 4a

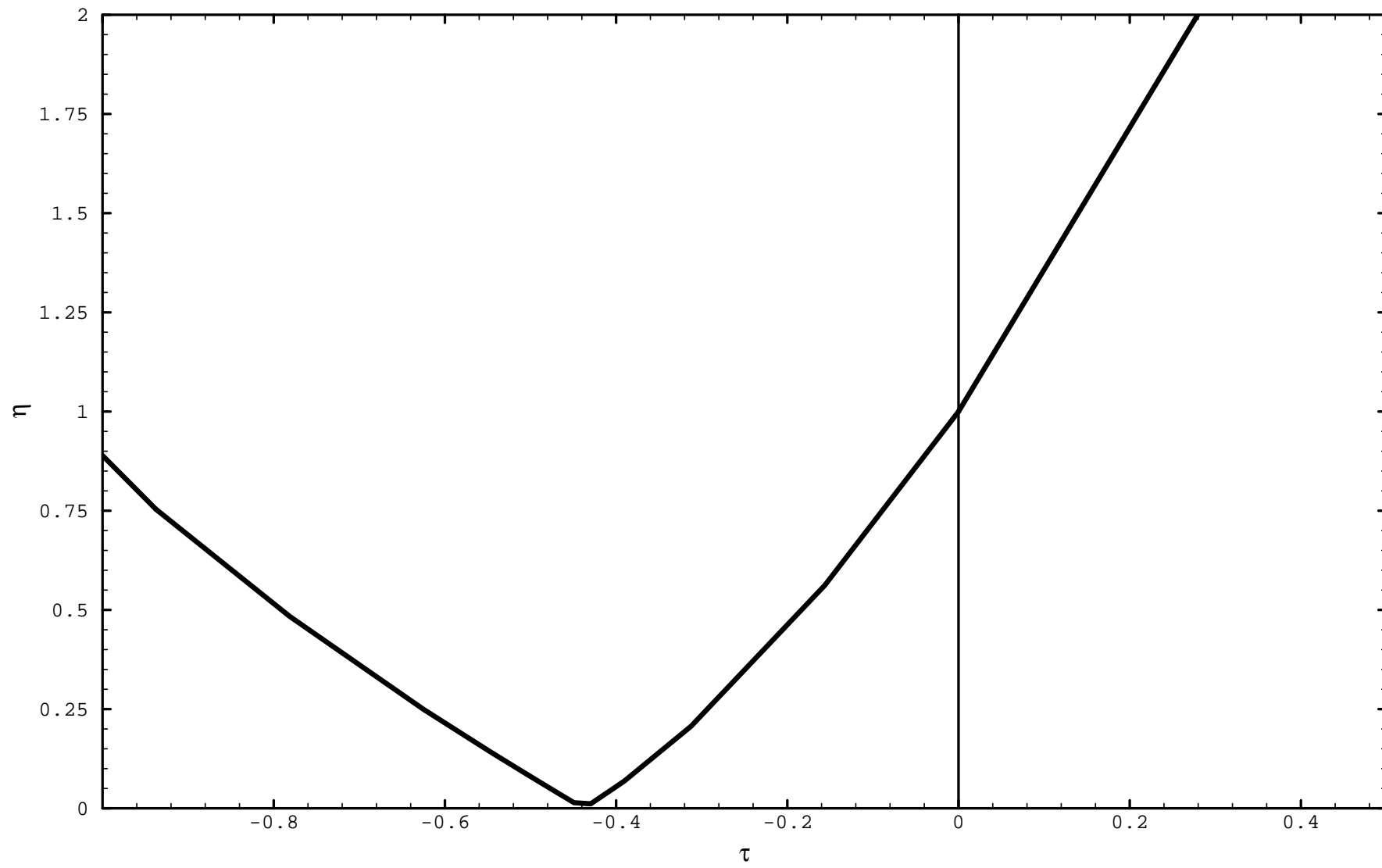


Fig. 4b

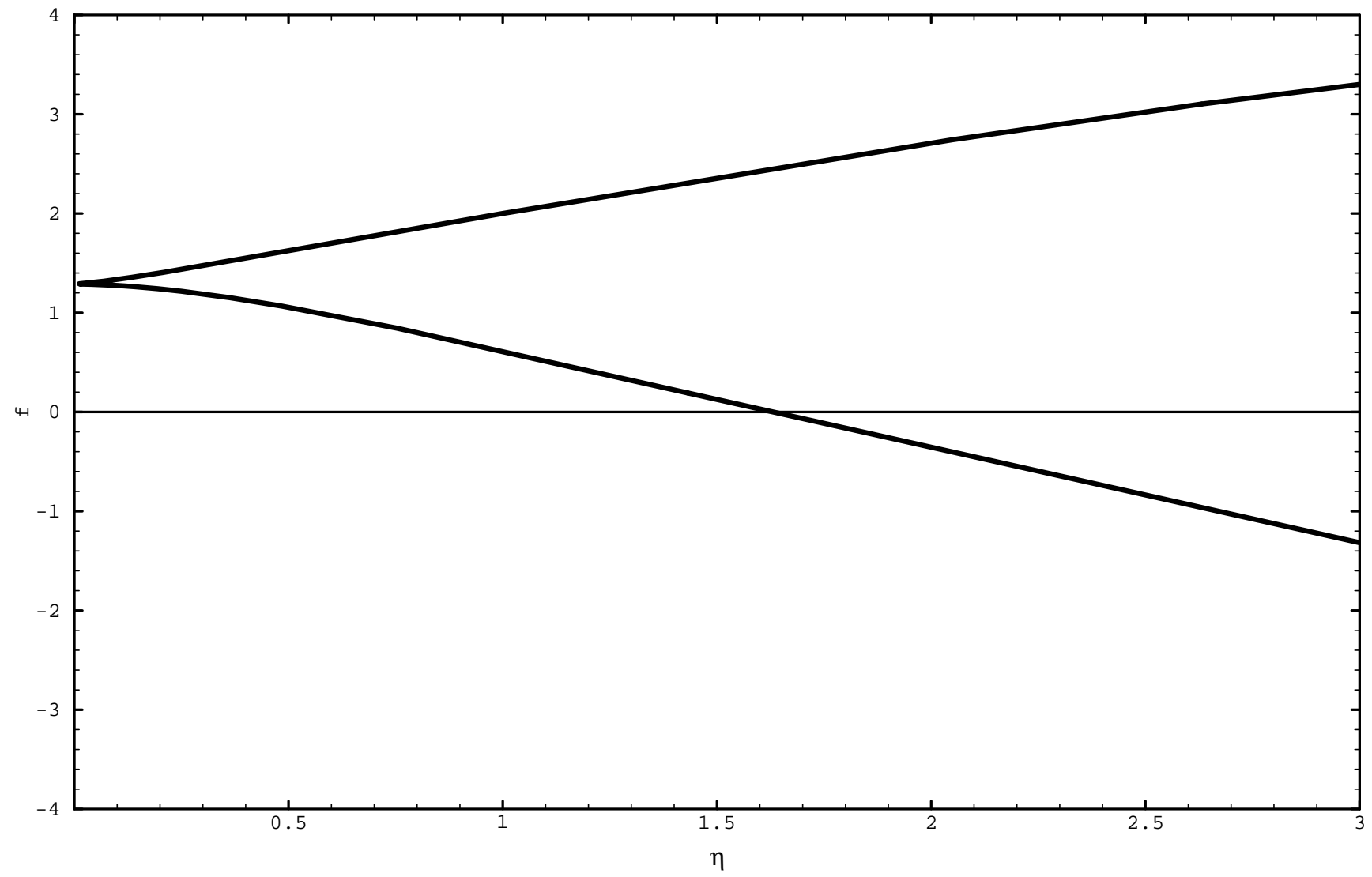


Fig. 4c

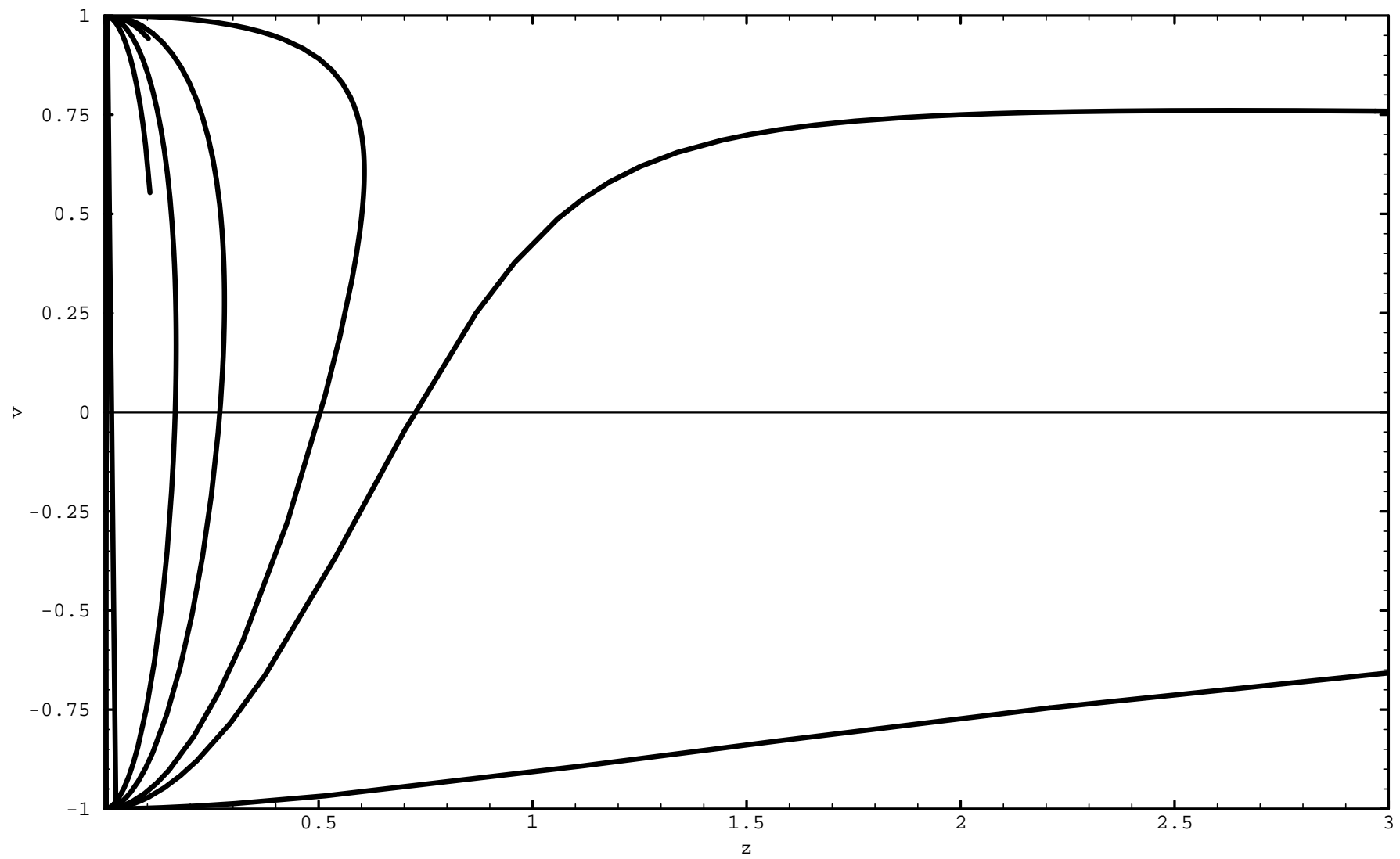


Fig. 5a

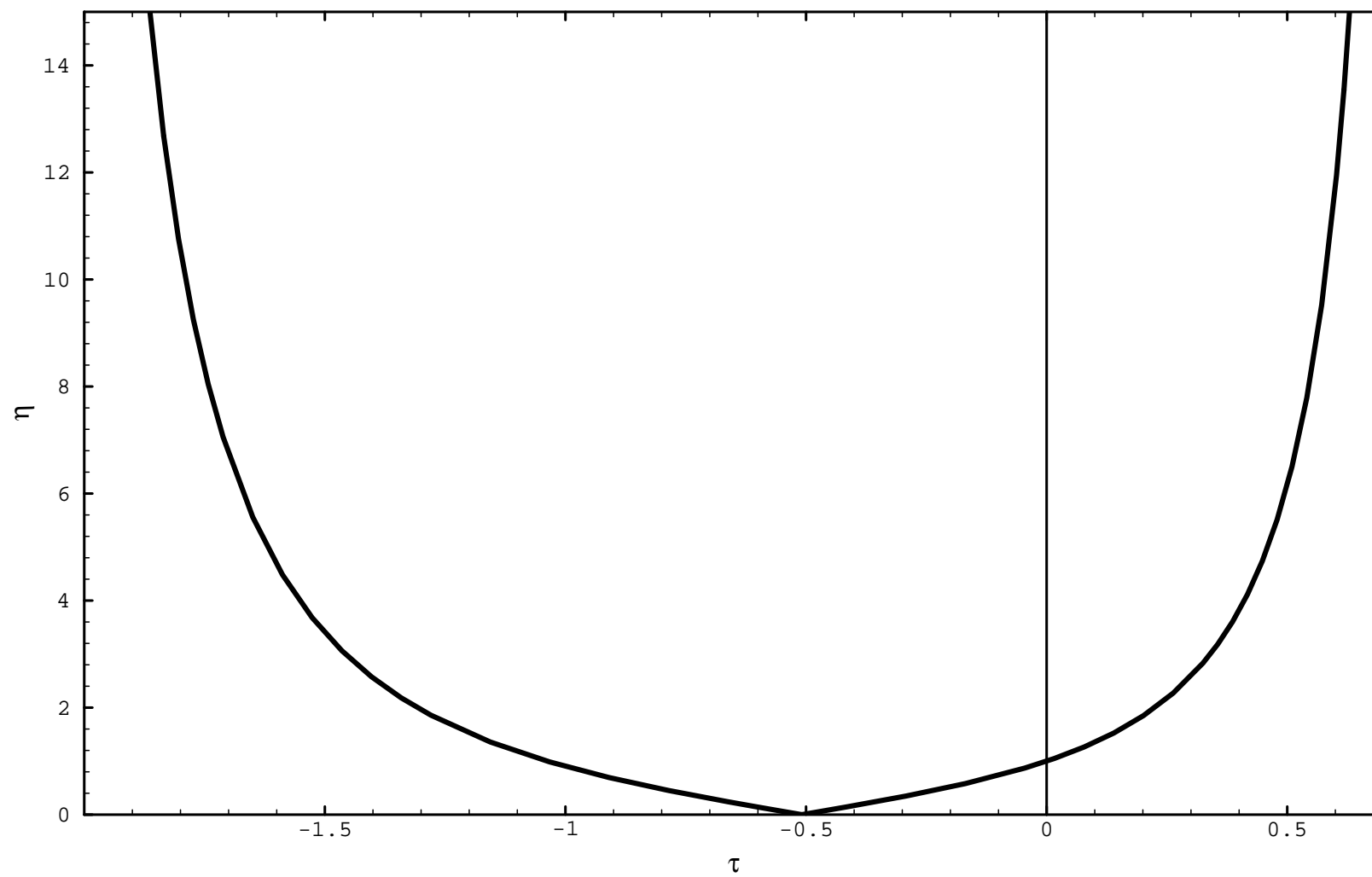


Fig. 5b

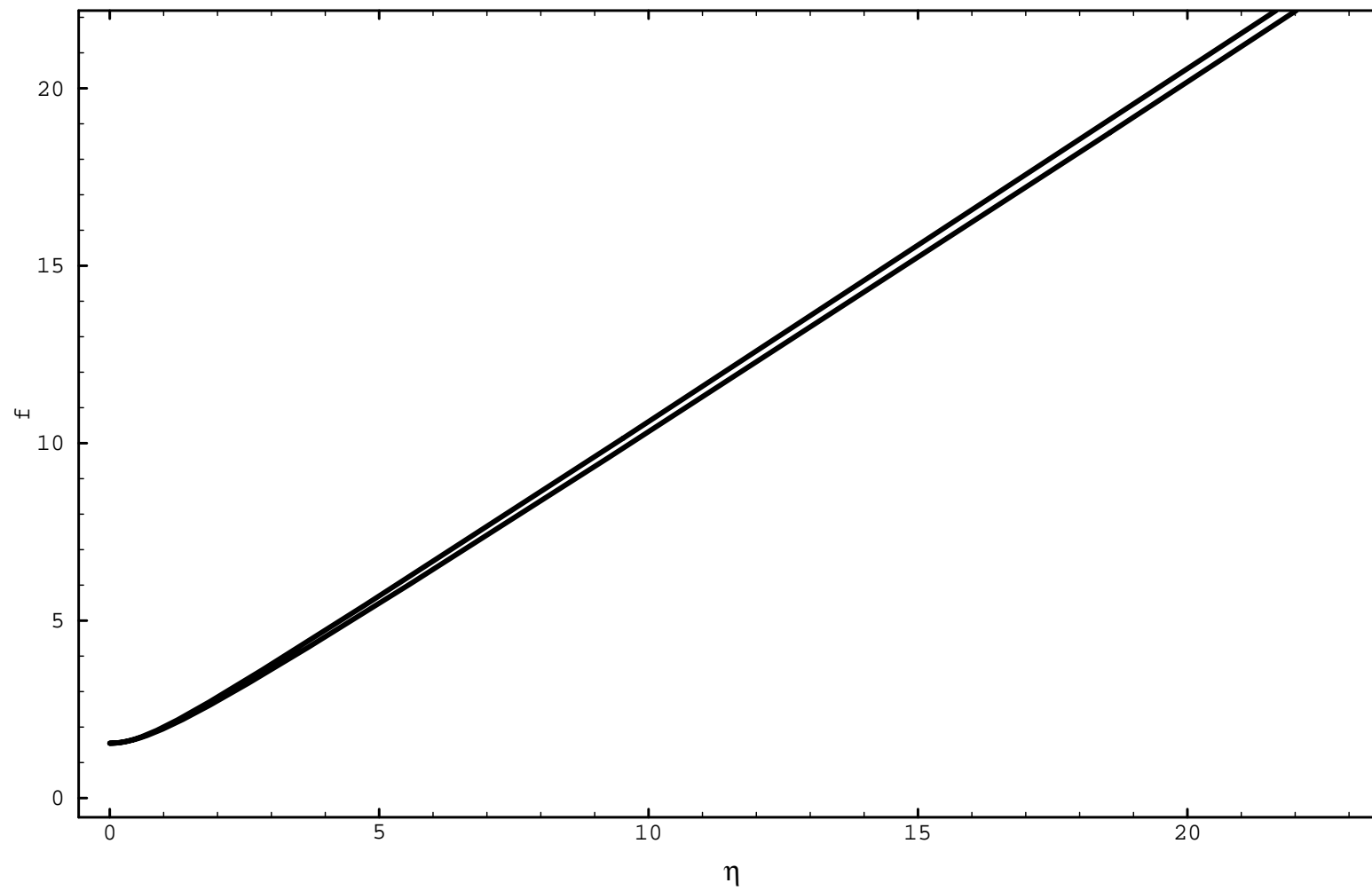


Fig. 5c

